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Self-interaction of a point charge in the Kerr space–time

B Léauté and B Linet

Equipe de Recherche Associée au CNRS No 533, Université Paris VI, Institut Henri Poincaré, 11 rue Pierre et Marie Curie, 75231 Paris Cedex 05, France

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Abstract. We determine the electric and magnetic self-field of a point charge at rest on the symmetry axis of the Kerr space–time in the coordinate system in which the metric describes locally a constant, static and homogeneous gravitational field. The result differs from that of a uniformly accelerated point charge in Minkowski space–time because of the influence of global boundary conditions. It follows that we can determine the induced self-force on the point charge.

1. Introduction

DeWitt and Brehme (1960) have investigated the influence of a gravitational field on the equation of motion of a point charge considered as a test particle with respect to the gravitational field. They have shown that a force involving a non-local function of the metric and the past history of the point charge occurred in addition to the usual radiation-damping term of Dirac (1938) (this calculation has been improved by Hobbs (1968)). This force has been determined by DeWitt and DeWitt (1964) for a point charge falling non-relativistically in a static weak gravitational field. It is not connected with the emission of electromagnetic radiation and it will be non-zero even if the point charge is at rest.

However, for a point charge at rest in a static space–time another approach enables us to determine the induced electrostatic self-force. One can directly calculate it by considering the global electrostatic potential determined as the solution of the Maxwell equations. This point of view has been adopted by Unruh (1976) for a point charge at rest within a spherical shell of matter, by Vilenkin (1979) for a point charge at a large distance from the horizon in the Schwarzschild space–time and by Smith and Will (1980) who calculated the self-force which would be measured by an instantaneously co-moving, freely falling observer at the position of the point charge. For this last case, Frolov and Zel'nikov (1981) have given the expression for the self-energy.

Recently, we have considered the same problem (Léauté and Linet 1981) in the coordinate system in which the Schwarzschild metric describes locally, in a small neighbourhood of the point charge, a constant, static and homogeneous gravitational field. This field is equivalent to an accelerated coordinate system in Minkowski space–time in the sense that the form of the Maxwell equations is the same in both cases. We have discussed the possible significance of the principle of equivalence taking into account the existence of this self-force due to the boundary conditions at infinity in the global space–time.

The purpose of this paper is to extend the results given above to the case of a stationary axisymmetric space–time, more exactly, the Kerr black hole. This is made

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possible because Léauté (1977) has determined, in algebraic form, the electromagnetic vector potential of a point charge at rest on the symmetry axis. In the coordinate system in which the Kerr metric describes locally a constant, static and homogeneous gravitational field, we shall deduce the expressions of the electromagnetic potential in a small neighbourhood of the position of the point charge. Then we shall see that we can define a finite electromagnetic self-field, at the position of the point charge, in a consistent way. Then we shall give the exact expression of the self-force on the point charge at rest on the symmetry axis of the Kerr space–time. We note that in the linearised Kerr metric, Beig (1973) has given the self-force within the framework of the DeWitt and Brehme formalism.

2. Accelerated observer in the Kerr space–time

The Kerr space–time is characterised by two parameters: M the mass and a the angular momentum per unit mass with $a \leqslant M$ in order to get a black hole. In the Boyer and Lindquist (1967) coordinates the Kerr metric is

$$\mathrm{d}s^2 = \left(1 - \frac{2Mr}{\Sigma}\right) \mathrm{d}t^2 - \frac{\Sigma}{\Delta} \mathrm{d}r^2 - \Sigma \, \mathrm{d}\theta^2 + \frac{4aMr \sin^2\theta}{\Sigma} \mathrm{d}t \, \mathrm{d}\varphi - \sin^2\theta \left(r^2 + a^2 + \frac{2a^2Mr}{\Sigma} \sin^2\theta\right) \mathrm{d}\varphi^2 \tag{1}$$

with $\Delta = r^2 - 2Mr + a^2$ and $\Sigma = r^2 + a^2 \cos^2\theta$. We have chosen units in which $c = 1$ and $G = 1$. We shall study electromagnetism only outside the outer horizon $r_+ = M + (M^2 - a^2)^{1/2}$.

We consider an observer moving along the world line (c) defined by $r = r_0$ with $r_0 > r_+$ and $\theta = 0$ in the Kerr metric (1). The orthonormal tetrad carried by the observer is chosen to be the locally non-rotating frame (LNRF) introduced by Bardeen *et al* (1972). It is easy to prove that in this way, we define at the position $r = r_0$ and $\theta = 0$ an accelerated non-rotating observer (Misner *et al* 1973).

The coordinate system (t, r, θ, φ) is not adapted for describing the Kerr metric on the symmetry axis. Therefore we introduce Cartesian-like coordinates (x^0, x^i) by the usual transformation

$$x^0 = t \qquad x^1 = r \sin\theta \cos\varphi \qquad x^2 = r \sin\theta \sin\varphi \qquad x^3 = r \cos\theta \tag{2}$$

in which the Kerr metric would be written in a well defined manner at the position $x^1 = 0$, $x^2 = 0$ and $x^3 = r_0$ corresponding to $r = r_0$ and $\theta = 0$. The components of the LNRF tetrad along the world line (c) in the coordinate system (2) will be

$$\underset{(0)}{e}{}^{\alpha} : (\Sigma_0^{1/2} \Delta_0^{-1/2}, 0, 0, 0) \qquad \underset{(1)}{e}{}^{\alpha} : (0, 0, 0, \Delta_0^{1/2} \Sigma_0^{-1/2}) \qquad \underset{(2)}{e}{}^{\alpha} : (0, r_0 \Sigma_0^{-1/2}, 0, 0) \qquad \underset{(3)}{e}{}^{\alpha} : (0, 0, r_0 \Sigma_0^{-1/2}, 0) \tag{3}$$

with $\Delta_0 = r_0^2 - 2Mr_0 + a^2$ and $\Sigma_0 = r_0^2 + a^2$.

The vector $\underset{(0)}{e}^{\alpha}$ coincides with the unit tangent vector of the world line (c). The acceleration vector of the observer is given by the formula (Misner *et al* 1973)

$$a^{\alpha} = \underset{(0)}{e}^{\beta} \nabla_{\beta} \underset{(0)}{e}^{\alpha}$$

which has one component only

$$g = \underset{(1)}{e}^{\alpha} a_{\alpha} = M(r_0^2 - a^2)\, \Sigma_0^{-3/2}\, \Delta_0^{-1/2}. \tag{4}$$

The coordinate system (y^0, y^i) for which the metric (1) takes the form

$$\mathrm{d}s^2 = (1+2gy^1)(\mathrm{d}y^0)^2 - (\mathrm{d}y^1)^2 - (\mathrm{d}y^2)^2 - (\mathrm{d}y^3)^2 \tag{5}$$

up to the second-order corrections in the coordinates y^i in a small neighbourhood of the line (c), can be related to (x^0, x^i) by the general transformation

$$x^{\mu} = x^{\mu}(y^0) + \underset{(i)}{e}^{\mu}\, y^i - \tfrac{1}{2}\Gamma^{\mu}_{\nu\lambda}|_{(c)}\, \underset{(i)}{e}^{\nu}\, \underset{(j)}{e}^{\lambda}\, y^i y^j \tag{6}$$

where y^0 is the parameter of the world line (c). The subscript (c) indicates the evaluation on the curve (c). Calculations, which we shall not reproduce here, lead to the following expressions for the transformation (6):

$$\begin{aligned}
x^0 &= \Sigma_0^{1/2}\, \Delta_0^{-1/2}\, y^0 \\
x^1 &= r_0\, \Sigma_0^{-1/2}\, y^2 + a^2\, \Delta_0^{1/2}\, \Sigma_0^{-2}\, y^1 y^2 \\
x^2 &= r_0\, \Sigma_0^{-1/2}\, y^3 + a^2\, \Delta_0^{1/2}\, \Sigma_0^{-2}\, y^1 y^3 \\
x^3 &= r_0 + \Delta_0^{1/2}\, \Sigma_0^{-1/2}\, y^1 + \tfrac{1}{2} M (r_0^2 - a^2)\, \Sigma_0^{-2} (y^1)^2 - M r_0^2\, \Sigma_0^{-2} [(y^2)^2 + (y^3)^2].
\end{aligned} \tag{7}$$

3. Electromagnetic potential for the accelerated observer

The Maxwell equations have the general form

$$\nabla_{\rho} F^{\rho\mu} = 4\pi J^{\mu} \qquad \text{and} \qquad F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \tag{8}$$

where A^{μ} is the electromagnetic potential and J^{μ} is the current density. We choose electrostatic units. In a Kerr black hole, Léauté (1977) has derived in algebraic form the components of the electromagnetic vector potential A_t and A_{φ} of a point charge q at $r = r_0$ with $r_0 > r_+$ on the symmetry axis. We have the following expressions

$$\begin{aligned}
A_t = \frac{q}{(r_0^2 + a^2)\Sigma} \bigg[& (r_0 r + a^2 \cos\theta) \left(M + \frac{(r-M)(r_0 - M) - (M^2 - a^2)\cos\theta}{R} \right) \\
& + a^2 (r - r_0 \cos\theta) \frac{(r-M) - (r_0 - M)\cos\theta}{R} \bigg]
\end{aligned}$$

$$\begin{aligned}
A_{\varphi} = -\frac{qa}{r_0^2 + a^2} \bigg\{ & \frac{\sin^2\theta}{\Sigma} \bigg[(r_0 r + a^2 \cos\theta) \left(M + \frac{(r-M)(r_0 - M) - (M^2 - a^2)\cos\theta}{R} \right) \\
& + a^2 (r - r_0 \cos\theta) \frac{(r-M) - (r_0 - M)\cos\theta}{R} \bigg] - R \\
& + (r - r_0 \cos\theta) \frac{(r-M) - (r_0 - M)\cos\theta}{R} - M(1 - \cos\theta) \bigg\}
\end{aligned} \tag{9}$$

with

$$R^2 = (r-M)^2 + (r_0-M)^2 - 2(r-M)(r_0-M)\cos\theta - (M^2-a^2)\sin^2\theta.$$

Now we must transform (9) into the coordinate system (y^0, y^i) introduced by the formulae (6) and (2). Omitting terms of order $(y^i)^2$ in the calculations, from (9) we deduce

$$\begin{aligned}
A_0 &= q\frac{1+\frac{1}{2}gy^1}{\varepsilon} + A_{ij}^{(0)}\frac{y^i y^j}{\varepsilon} - q\frac{Mr_0}{(r_0^2+a^2)^2}y^1 \\
A_1 &= A_{ij}^{(1)}\frac{y^i y^j}{\varepsilon} \\
A_2 &= A_{ij}^{(2)}\frac{y^i y^j}{\varepsilon} + \frac{1}{2}q\frac{Ma}{(r_0^2+a^2)^2}y^3 \\
A_3 &= A_{ij}^{(3)}\frac{y^i y^j}{\varepsilon} - \frac{1}{2}q\frac{Ma}{(r_0^2+a^2)^2}y^2
\end{aligned} \tag{10}$$

where $\varepsilon = [(y^1)^2 + (y^2)^2 + (y^3)^2]^{1/2}$. We can leave the constant quantities $A_{ij}^{(\mu)}$ undetermined as they will not be needed in the following discussion.

Using a standard procedure we can define at the position of the point charge an electric self-field and a magnetic self-field in the coordinate system (y^0, y^i). We introduce the usual polar angles $\Omega = (\alpha, \beta)$ related to y^i and we set

$$\begin{aligned}
E^i_{\text{self}} &= \lim_{\varepsilon_0\to 0}\frac{1}{4\pi}\int_{\varepsilon=\varepsilon_0} F_{0i}\,d\Omega \\
B^i_{\text{self}} &= \lim_{\varepsilon_0\to 0}-\frac{1}{4\pi}\int_{\varepsilon=\varepsilon_0} F_{jk}\,d\Omega \qquad i, j, k \text{ circular permutation.}
\end{aligned} \tag{11}$$

The only term in the expressions (10) giving an infinite field in the formulae (11) is the first term in the electrostatic potential A_0. We recall that this term corresponds to the electromagnetic potential of a point charge which is uniformly accelerated in the Minkowski space–time (Rohrlich 1963). The terms in (10) including the quantities $A_{ij}^{(\mu)}$ do not contribute to the self-field. Finally the last terms in (10) give the following finite self-field:

$$\boldsymbol{E}^i_{\text{self}} = q\frac{Mr_0}{(r_0^2+a^2)^2}\delta^i_1 \qquad \boldsymbol{B}^i_{\text{self}} = q\frac{Ma}{(r_0^2+a^2)^2}\delta^i_1. \tag{12}$$

4. Concluding remarks

The results presented in this paper extend our previous work (Léauté and Linet 1981) to the case of the Kerr black hole. Compared with the case of the Schwarzschild black hole, there is now also a magnetic self-field besides the electric self-field. In order to calculate them we have used the global solution of the Maxwell equations. The electromagnetic self-field (12), induced specifically by the gravitational field, results from the influence of global boundary conditions.

Only the electric self-field exerts a self-force on the point charge at rest on the symmetry axis. Moreover we have not to consider the infinite force arising from the first term in (10) because it can be absorbed in the mass renormalisation as in a Minkowski space–time. From (12) we find, always in the coordinate system (y^0, y^i),

$$f^i_{\rm self} = q^2 \frac{Mr_0}{(r_0^2 + a^2)^2} \delta^i_1. \tag{13}$$

In the linearised Kerr metric, which is the Lense–Thirring metric, we note that (13) reduces to the Schwarzschild value.

Furthermore if the point charge has a magnetic dipole moment, then the magnetic self-field produces a self-torque as suggested by Parker (1981) for a point charge in free fall.

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